No.

Date

Radon-Nikodym Theorem

Measure don't need to be positive! $\Rightarrow$ Signed measure $u: M \longrightarrow \mathbb{R}$ (finite hence we don't need to deal with $(-\infty)+(\infty)$) $\Rightarrow$ Total variation induced by signed measurable

Def. (Signed measure) M is a σ-algebra on X.

$u: M \longrightarrow \mathbb{R}$ is a signed measure if it satisfies:

$u(E) = \sum_{i=1}^{\infty} u(E_j)$, given $E = \bigcup_{j=1}^{\infty} E_j$.

Remark: $u(X) < \infty$; $u(\phi) = 0$

Given a signed measure, we can get a total variation by:

$|u|(E) = \sup \{ \sum_{j=1}^{\infty} |u(E_j)| \}$

prop: TV is a finite measure on (X, M).

proof: • TV satisfies countable addivity.

• $|u|(X)$ is finite:

[Lemma: If $|u|(E) = \infty$ for some $E \in M$, then $\exists$ $A, B \in E$, $E = A \cup B$, and $|u(A)|, |u(B)| \geq 1$].

Now $|u|(X) = \infty$. we beak X into A and B, $|u(A_1)|, |u(B_1)| \geq 1$.

$|u|(X) = |u|(A_1) + |u|(B_1) \Rightarrow$ WLOG suppose $|u|(A_1) = \infty$.

Using A_1 to replace X. $\exists$ $A_2 . B_2$ st $|u|(A_1) = |u|(A_2) + |u|(B_2)$

$B_{n+1} \subseteq A_n$, $\{B_i\}$'s are disjoint. $B = \bigcup_{j=1}^{\infty} B_j$

$u(B) = \sum_{j=1}^{\infty} u(B_j)$.

u is a signed measure, $u(B_j) \longrightarrow 0$ as $j \to \infty$.

But we suppose $|u(B_j)| \geq 1$, contradiction!

Example of $\mathcal{L}^1$-functions:

u is a measure on (X, M) and $f \in L^1(u)$, then:

(a) $\lambda(E) = \int_E f du$, $\forall E \in M$ is a signed measure.

(b). $|\lambda|(E) = \int_E |f| du$ is the induced total variation.

KOKUYO

No.

Date

We discuss more about "algebraic" properties of the set of all signed measures on $(X, \mathcal{M})$.

- It's a vector space. it's complete under the norm: $\|u\| = |u|(X)$
- given a signed measure u. let $u^+ = \frac{1}{2}(|u|+u)$, $u^- = \frac{1}{2}(|u|-u)$, $u = u^+ - u^-$, $|u| = u^+ + u^-$, this is called the Jordan decomposition of the signed measure.

(DEF) absolute continuous: λ. u are two measures. λ is absolutely continuous w.r.t u. $\lambda \ll u$, if every u-null set is a λ-null set.

concentrate: λ is concentrate on set A if $\lambda(E) = \lambda(E \cap A)$. $\forall E \in \mathcal{M}$

singular to each other: λ_1 and λ_2 are singular to each other. $A \cap B = \phi$, λ_1, λ_2 concentrate on A, B respectively, they say λ_1 is singular to λ_2. $\lambda_1 \perp \lambda_2$.

Let u be a measure, λ a measure or signed measure on $\mathcal{M}$.

(prop):
(a) λ is concentrate on $A \Rightarrow |\lambda|$ is concentrate on A
(b) $\lambda_1 \perp \lambda_2 \Rightarrow |\lambda_1| \perp |\lambda_2|$
(c). $\lambda_1 \perp u$. $\lambda_2 \perp u \Rightarrow \lambda_1 + \lambda_2 \perp u$
(d). $\lambda_1 \ll u$. $\lambda_2 \ll u \Rightarrow \lambda_1 + \lambda_2 \ll u$
(e). $\lambda \ll u$, $\Rightarrow |\lambda| \ll u$
(f). $\lambda_1 \ll u$. $\lambda_2 \perp u \Rightarrow \lambda_1 \perp \lambda_2$
(g). $\lambda \ll u$, $\lambda \perp u \Rightarrow \lambda = 0$

It seems that absolute continuity and singularity are 2 extreme relations between 2 measures. However we are suprised to find. they can be find (almost) in every measure, every u can be split in to "absolute continuous" part and "singular" part.

Theorem (Lebesgue Decomposition)

Let u be a σ-finite measure and λ a signed measure on (X, M), we can split λ into $\lambda_{ac} + \lambda_s$, where $\lambda_{ac} \ll u$ and $\lambda_s \perp u$. this decomposition $\lambda = \lambda_{ac} + \lambda_s$ is unique

Theorem (Radon–Nikodym Theorem)

u is a σ-finite measure. λ a signed measure on (X, M), s.t. $\lambda \ll u$. There exists a unique $h \in L^1(u)$. st. $\lambda(E) = \int_E h\,du$. $\forall E \in M$

The function h is called the Radon–Nikodym derivative of λ w.r.t u and will be denoted by $\frac{d\lambda}{du}$.

Proof: ① u and λ are finite measures

- Let $\rho = u + \lambda$. def $\Lambda \varphi = \int \varphi\, d\lambda$. $\varphi \in L^2(\rho)$

  $|\Lambda \varphi| \le \int |\varphi| d\lambda \le \sqrt{\rho(X)}\, \|\varphi\|_{L^2(\rho)}$

- Λ is a bounded linear functional on $L^2(\rho)$. Thus By the self-duality of $L^2(\rho)$,

  $\exists g \in L^2(\rho)$, s.t. $\int \varphi\, d\lambda = \int \varphi g\, d\rho$. $\forall \varphi \in L^2(\rho)$

- Let $\varphi = \chi_E$. $\int_E d\lambda = \lambda(E) = \int_E g\, d\rho$. $\frac{\lambda(E)}{\rho(E)} = \frac{\int_E g\, d\rho}{\rho(E)} \in [0,1]$

  and $g \in [0,1]$ ρ-a.e. We can assume $g(X) \subseteq [0,1]$.

  $\lambda_{ac}(E) = \lambda(E \cap \{x : g(x) \in [0,1)\}) :\doteq \lambda(E \cap A)$

  $\lambda_s(E) = \lambda(E \cap \{x : g(x) = 1\}) := \lambda(E \cap B)$

  $\int \varphi(1-g)\, d\lambda = \int \varphi g\, du$. Let $\varphi = \chi_B$. then $u(B) = 0 \Rightarrow \lambda_s \perp u$

  Let $\varphi = \chi_E(1 + g + g^2 + \cdots + g^n)$. $\int \chi_E(1 - g^{n+1})\, d\lambda = \int \chi_E(1 + g + \cdots + g^n) g\, du$

  $\downarrow$

  $\int_{E \cap A} (1 - g^{n+1})\, d\lambda \to \lambda(E \cap A) = \lambda_{ac}(E)$.

  $RHS = \int_{E \cap A} g \frac{1 - g^{n+1}}{1-g}\, du \to \int_{E \cap A} \frac{g}{1-g}\, du$

  $\lambda_{ac}(E) = \lambda(E \cap A) = \int_{E \cap A} h\, du$, $h = \frac{g}{1-g}$, $\lambda_{ac} \ll u$

②. u is finite measure. λ signed measure

$\lambda^+ = \lambda_{ac}^+ + \lambda_s^+$. $\lambda^- = \lambda_{ac}^- + \lambda_s^-$.

$\lambda_{ac}^+, \lambda_{ac}^- \ll u$. $\lambda_{as}^+, \lambda_s^- \perp u$

No.

Date

③ μ is σ-finite. λ is signed measurable:

$X = \bigcup_{j=1}^{\infty} X_j$, $\mu(X_j) < \infty$.

$\lambda_j = \lambda_{ac}^j + \lambda_s^j$, $\lambda_{ac}^j \ll \mu_j$, $\lambda_s^j \perp \mu_j$.

then $\lambda_{ac} = \sum_{j=1}^{\infty} \lambda_{ac}^j$, $\lambda_s = \sum_{j=1}^{\infty} \lambda_s^j$, $\lambda = \lambda_{ac} + \lambda_s$. $\lambda_{ac} \ll \mu$, $\lambda_s \perp \mu$

④ Using contradition to prove uniqueness.

A significant Application:

μ is a signed measure on (X, M), $|\mu(E)| \leq |\mu|(E)$, $\Rightarrow \mu \ll |\mu|$. By R-N theorem, $\exists\ h \in L^1(|\mu|)$, st $\mu(E) = \int_E h \, d|\mu|$

$$\frac{1}{|\mu|(E)} \cdot \int_E h \, d|\mu| = \frac{\mu(E)}{|\mu|(E)} \in [-1, 1] \Rightarrow h \leq 1 \text{ a.e.}$$

Claim: $|h| = 1$ a.e.

$A_r = \{x \in X : |h|(x) < r\}$. $r \in (0,1)$.

$$\sum_j |\mu(A_j)| = \sum_j \left| \int_{A_j} h \, d|\mu| \right| \leq \sum_j \int_{A_j} |h| \, d|\mu| \leq r \sum_j |\mu|(A_j) = r|\mu|(A_r)$$

Take supremum over all partitions $\{A_j\}$.

$|\mu|(A_r) \leq r|\mu|(A_r)$

As $|\mu|(A_r) \leq |\mu|(X) < \infty$. $\Rightarrow A_r$ has $|\mu|$-measure 0.

$\Rightarrow |h| = 1$ $|\mu|$-a.e.. After redefining h on a set of measure 0, we may assume $|h| = 1$ everywhere.

Conclusion: μ is a signed measure on (X, M).

(a) $\exists\ h \in L^1(|\mu|)$, $|h| \equiv 1$, s.t. $d\mu = h \, d|\mu|$.

(b). $\exists$ disjoint measurable sets A and B, $\mu^+(E) = \mu(E \cap A)$; $\mu^-(E) = -\mu(E \cap B)$. $\forall E \in M$

(c). $\mu = \lambda_1 - \lambda_2$, ($\lambda_i$ are measures), then $\lambda_1 \geq \mu^+$, $\lambda_2 \geq \mu^-$